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STRUCTURAL DIVISION

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ANALYSIS OF ARCHES BY FINITE DIFFERENCES

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SYNOPSIS

In flexible arches vertical and horizontal displacements of the rib due to imposed loading cause large increases in the bending moments computed on the basis of invariant arch geometry. These increases in moments cannot be neglected in safe design. Present methods for solving such problems are either restrictive or are cumbersome in application. The present paper is an attempt to remedy this situation. Thus, starting with two basic differential equations which relate both the vertical and horizontal deflections of the rib to the internal forces, a solution of these equations by the finite difference method is discussed. Arches of any geometry, cross-section, and conditions of loading may be analyzed by the proposed method. Moreover, the proposed method is rapid in applications, and, due to its simplicity, may be mastered readily.

NOTATION

The following list contains the most important of the symbols used in this paper.

A	cross-sectional area
E	modulus of elasticity
f	maximum rise of arch
H	horizontal thrust
h	length of finite difference interval
I	moment of inertia
ℓ	span length = n h
M	bending moment of arch rib
M'	simple beam bending moment
M_o	fixed end moment
M_ξ	bending moment due to horizontal displacement
M_η	bending moment due to vertical displacement = $H\eta$
N	axial thrust
n	number of finite difference intervals

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t	change of temperature
$V_{L,R}$	left, right vertical reactions
$w(x)$	transverse loading
w_d	dead load, load per unit length
wl	live load, load per unit length
x, y	Cartesian co-ordinates defining original shape of arch axis
α	coefficient of thermal expansion
η	vertical deflection of arch axis
θ	slope of arch axis
ξ	horizontal deflection of arch axis

INTRODUCTION

There are two basic theories that may be employed in the analysis of arch bridges. The classical theory of analysis, often known as the "elastic" theory, assumes that any deflections that occur in the rib due to external loading are so small as to have negligible effect upon the internal bending moments. Such an approach is suited only to the analysis of very rigid masonry or concrete arches. However, the design of flexible arches requires consideration of the changes in geometry of the rib that occur when external loadings are imposed upon the structure. It is no longer possible in these cases to consider that the arch has an invariant geometry, for the changes in geometry, though small in comparison with the co-ordinates of the arch, in some cases cause bending moments that are much larger than those computed by elastic theory.

Consider the two-hinged arch shown in Fig. 1 which is acted upon by some arbitrary vertical loading $w(x)$. The resultant reactions at the supports will be V_R and V_L vertically and a horizontal thrust H. A point A on the rib will displace η in a vertical direction and ξ in a horizontal direction to a new position A'. The co-ordinates of point A are x and y in the undistorted position of the arch and $x + \xi$ and $y + \eta$ in the distorted position A'. In the elastic theory it is considered that the bending moment at any point on the arch rib is the difference between the bending moment that would exist if the span were a simple beam and the moment caused by the product of the horizontal thrust and the initial ordinate of the point in question. However, it can be seen that any horizontal deflections will change the lever arms of the vertical loads and thus will change the value of the simple beam bending moment. Moreover, any vertical deflection will change the lever arm of the horizontal thrust. Now, while the numerical values of the simple beam moment and the moment caused by the horizontal thrust are in themselves quite large, the magnitude of their difference becomes of the same order as the products $\overline{H\eta}$, $\overline{V\xi}$, and $\overline{w(x)\xi}$. Small deflections, therefore, can drastically change the net design moments. In such cases it is necessary to employ an analysis that will include the effects of deflection in the solution of the problem. This type of analysis is known as the "deflection" theory of arch bridges. For the correct and safe design of large span, flexible arches, the deflection theory is indispensible and it does not represent a mere refinement of the elastic theory.

Initial investigations of the deflection theory of arches were made in a

series of papers[1,2,3,4,5][3] published by Josef Melan and Fritz Engesser between 1900 and 1906. In 1934 B. Fritz[6] extended Melan's work to cover three-hinged, two-hinged or hingeless arches. In a theory developed in 1931, S. Kasarnowsky[7] presented a simple calculational procedure. In 1935, A. Freudenthal[8,9] proposed a theory for a two-hinged or hingeless arch in which the variable moment of inertia I was equal to $I_c \sec \theta$, where I_c is moment of inertia at the crown.

In all of the above procedures the horizontal component of deflection is neglected. This is a satisfactory assumption only for very flat arches. In addition, all of the enumerated procedures are restricted to arches which have a definitely prescribed variation of the cross-section or, except for the work of Fritz, have restrictions as to the placement of live loading upon the structure. To remedy the situation, G. Stern[10] developed a method utilizing a Fourier series solution for a two-hinged arch and was able to solve the problem for any variation of arch rib or loading. However, the horizontal component of deflection was still neglected. This work was extended by W. F. Lu[11] in 1951 to a hingeless arch. A simple approximate procedure by use of a magnification factor was developed by Hardesty et al[12] in conjunction with the Rainbow Arch design in 1945. The same point of view was also used[13] by R. S. Rowe.

In 1948, P. N. Chatterjee[14] developed a method for the analysis of two-hinged arches by use of the Newmark numerical procedure. The advantages of his work are that it may be applied to an arch with any variation of rib properties and live loading and, in addition, it accounts for the horizontal component of deflection. This work was checked by means of a model study by C. L. Monismith[15] in 1954.

Any attempt to solve the deflection theory of arches for a general case of arch rib properties and live loading presents an almost intractable mathematical problem unless some sort of numerical procedure is employed. One such approach is Chatterjee's solution, cited above. It is, however, a successive approximation type solution and requires several cycles of computation to obtain the final results. It is proposed in this paper to demonstrate that the use of the finite difference method of solving differential equations allows the rapid and accurate analysis of any arch structure regardless of its geometry, cross-sectional variation or external loading. It is furthermore applicable to either the elastic theory or deflection theory of arch action. The special steps and precautions required in the application of this method to the arch will be discussed first, and then, in order to aid in clarification of this discussion, numerical problems for both the elastic and deflection theories will be presented.

Differential Equations of the Arch

The action of arch ribs is governed by two differential equations which relate the vertical and horizontal deflections of the rib to the internal forces in the rib. The two equations are not widely known and to the best of the authors' knowledge, have not been cited previously in civil engineering literature. These equations were developed[16] and are stated here in terms of the horizontal and vertical deflections, rather than in terms of radial displacements. The two equations are:

3. Numbers in parentheses refer to the Bibliography at the end of the paper.

$$\frac{d^2\eta}{dx^2} = \frac{M}{EI}\sec\theta - \left(\frac{N}{EA} - \alpha t\right)\frac{d^2y}{dx^2} \quad (1)$$

$$\frac{d\xi}{dx} = -\frac{d\eta}{dx}\tan\theta + \left(\frac{N}{EA} - \alpha t\right)\left(1 - \tan^2\theta\right) \quad (2)$$

The first term on the right-hand side of Eq. 1 is due to flexure while the second term is due to axial deformation. If flexural effects alone are to be considered, then this latter term is omitted. Note that if the effects of axial deformation are neglected and thus only the first term on the right-hand side of Eq. 2 is retained, the equation becomes one which relates the rate of change of vertical deflection to rate of change of horizontal deflection. This equation is then in effect a "coupling" equation.

Solution of the Arch by Finite Differences

The Basic Procedure

For any given arch and condition of loading, it is necessary to solve the basic Eqs. 1 and 2 for the deflection functions $\xi(x)$ and $\eta(x)$ in order to determine the distorted geometry of the arch. In addition, the redundant reactions must be determined. It is then possible, as may be seen from Fig. 1, to evaluate by statics the internal forces at any section of the arch rib.

In this paper it is suggested that the solution be accomplished by the use of finite differences. The characteristic of this procedure is that the differential equations are transformed into a series of algebraic equations. To accomplish this, approximate relations for the derivatives are used. For example, in the "three-point-central difference" approximation, the first and second derivatives of a function f(x) at an m-th point may be expressed in terms of the values of the function at the adjoining points as follows:[17]

$$f'(x)_m \doteq \frac{1}{2h}\left[f(x)_{m+1} - f(x)_{m-1}\right] \quad (3)$$

$$f''(x)_m \doteq \frac{1}{h^2}\left[f(x)_{m-1} - 2f(x)_m + f(x)_{m+1}\right] \quad (4)$$

To express higher order derivatives it is necessary to use five or more adjoining points because of the restrictions inherent in the derivation of Eqs. 3 and 4. However, since the arch differential equations are not higher than second order, the three-point approximation is sufficient.

To apply the method of finite differences, the span of the structure to be analyzed is divided into any desired number of segments. For ease of solution, it is usually advisable to have segments of equal length, although a mesh of variable spacing may be used to advantage if the behavior of a particular point on the arch needs to be investigated more accurately. In any event, the degree of accuracy of the solution is dependent upon the fineness of the mesh chosen. Excellent results were obtained for the numerical examples of this paper by using a mesh of only 8 subdivisions.

At each of the interior points obtained by this subdivision the differential equations are converted to algebraic equations by using the finite difference

expressions for first and second derivatives and by substituting the values at each point of the physical constants occurring in these equations. On this basis, if the arch span is divided into n segments, (n - 1) algebraic equations are obtained for each of the basic differential equations. However, the set of algebraic equations formed from Eq. 1 will contain n unknowns: an unknown vertical deflection, η_m, at each interior point, and the horizontal thrust, H. It is, therefore, necessary to eliminate one unknown from this set of algebraic equations by the use of Eq. 2 and the boundary conditions of the problem.

The Boundary Conditions

For a pin-ended arch, since the moments at the ends are zero, the boundary conditions in finite difference form become:

$$\left.\begin{aligned} \eta_{-1} &= -\eta_1 \;;\; \xi_{-1} = -\xi_1 && \text{at left springing} \\ \eta_{n-1} &= -\eta_{n+1} \;;\; \xi_{n-1} = -\xi_{n+1} && \text{at right springing} \end{aligned}\right\} \quad (5)$$

where the subscript -1 signifies an imaginary point one interval h to the left of the left hinge. The subscript n+1 signifies an analogous point to the right of the right end.

For a hingeless arch, the boundary conditions are:

$$\left.\begin{aligned} \eta_{-1} &= +\eta_1 \;;\; \xi_{-1} = +\xi_1 && \text{at left springing} \\ \eta_{n-1} &= \eta_{n+1} \;;\; \xi_{n-1} = \xi_{n+1} && \text{at right springing} \end{aligned}\right\} \quad (6)$$

These boundary conditions may be used in the case of a hingeless arch to define the redundant end moments. Thus, writing Eq. 1 in finite difference form at the left springing and neglecting axial distortions, the following relation is obtained:

$$\frac{1}{h^2}\left(\eta_{-1} - 2\eta_0 + \eta_1\right) = \frac{M_0}{(EI)_0} \sec\theta_0$$

and since $\eta_0 = 0$, and $\eta_{-1} = \eta_1$,

$$M_0 = \frac{(EI)_0}{h^2 \sec\theta_0} 2\eta_1 \quad (7)$$

A similar expression may be formulated for the moment at the right springing of an arch and these expressions may be used in lieu of the redundant unknown moments in the solution of a problem.

When Eq. 2 is applied in finite difference form at the boundary points in addition to the interior points of the subdivision, it becomes possible to express vertical displacement in terms of each other. This permits the elimination of one of the superfluous η' s from the algebraic equations based on Eq. 1. In order to avoid a trivial relationship of zero equals zero, for hingeless arches it is first necessary to differentiate Eq. 2 once with respect to x before writing it in finite difference form at the boundary points. In addition, Eq. 2

may be used to define all horizontal displacements in terms of the vertical displacements. This is necessary to eliminate the ξ terms that appear in Eq. 1 when horizontal displacements are considered in the analysis.

Procedure for the Elastic Theory

In elastic theory the object of the analysis is the determination of the redundant forces. The analysis is not concerned with the deflections of the arch per se. Although it recognizes that the deflections exist, it assumes that they have no effect upon the final stresses. For a two-hinged arch, the algebraic equations, the formulation of which was discussed above, can be solved for the value of the horizontal thrust H. For a hingeless arch, it is also necessary to determine the vertical deflections occurring at each of the stations adjacent to the two springings. These deflections may then be used to determine the two redundant end moments by means of Eq. 7.

Procedure for the Deflection Theory

If an arch is to be analysed by the deflection theory, the expressions for internal bending moment must first be modified to account for the change in geometry of the rib. It was indicated in the Introduction that vertical deflections affect the moment caused by horizontal thrust and horizontal deflections change the moments caused by vertical loading. The lever arm for the thrust is modified at each station by expressing it as the ordinate, y, at the station plus the as yet unknown vertical deflection, η, at that station. In algebraic form, this changes the expression for bending moment to:

$$M = M' - H(y + \eta) \tag{8}$$

No such general rule is possible when dealing with the horizontal displacements. For general cases of transverse loading, it is necessary to formulate the primary bending moment at each point by statics, considering that each point has a positive horizontal displacement ξ. For the special case of an arch carrying a uniformly distributed vertical load, however, it is possible to obtain a reasonably accurate general expression for the primary bending moment modified for the change in geometry. Thus, consider the arch shown in Fig. 1 which carries a partial uniform vertical load w causing the vertical reactions V_L and V_R. These reactions will be assumed to be those for a simple beam carrying the indicated transverse load. The simple beam moment about A' will then be

$$M'_\xi = M' + M_\xi \doteq V_L(x + \xi_A) - (w)(x)\left(\frac{x}{2} + \xi_A\right)$$

$$\doteq M' + \xi_A(V_L - wx) \tag{9}$$

This expression holds for situations where the uniform load extends from the springing at least to the point in question. A similar expression can be formulated if the load extends from the right-hand springing.

Note that the consideration of vertical deflections gives rise to terms which are the products of the horizontal thrust and the vertical deflection at each point. Both these quantities are unknown and it is desirable that one of them be intelligently estimated in order to permit the direct solution of the matrix of the algebraic equations based on Eq. 1. The effect that deflections have upon the value of horizontal thrust is usually relatively small. Hence, the arch may be solved initially for the value of this redundant force by the elastic theory and the value of this thrust then be used in the deflection theory analysis.

The above modifications in the expression for the bending moments are all that are required for the deflection theory solution. If horizontal deflections are not to be considered, then the change in the lever arm of the horizontal thrust alone need be considered. In addition, since the thrust is assigned an assumed numerical value, the only unknowns that then appear are the various vertical deflections. If horizontal deflections are considered, it is necessary to eliminate the additional unknowns by the methods described previously.

Upon solving the algebraic equations for the various deflections, the bending moments in the arch may be determined by the substitution of these values into the previously formulated Eqs. 8 and 9. The true bending moments are thus found directly from the initial solution and several successive approximations are not required. However, initially the value of thrust based on the elastic theory was assumed and does not exactly correspond with its true value. It is likely that any difference between the two values will be insignificant. If it is desired to solve the equations by an iterative procedure rather than by the direct solution suggested here, the H may be left initially as an unknown. Such a procedure may be carried out with the aid of electronic computers.

It should be noted that if for symmetrical loading, the "three-point central difference" approximation is used in establishing the finite difference equations the apparent result is that equal horizontal displacements occur for the first point on each side of the crown. This contradicts the actual situation of equal and opposite horizontal displacements at these two points. This contradiction can be eliminated by using the "five-point central difference" approximation in setting up the finite difference equations. Alternatively, with the "three-point" approximation the finite difference equations may be written from both ends of the arch toward the crown. Then a relation for the horizontal displacements at the crown point need not be used at all and no contradiction occurs.

Numerical Examples

The two-hinged arch shown in Fig. 2 will be solved in this section by both the elastic and deflection theories. In both solutions the effects of axial distortion will be neglected in order to obtain values which can be compared with other known analyses. For this purpose, the differential Eqs. 1 and 2 simplify to

$$\frac{d^2\eta}{dx^2} = \frac{M}{EI} \sec\theta \qquad \text{(1a)}$$

and

$$\frac{d\xi}{dx} = -\frac{d\eta}{dx} \tan\theta \qquad \text{(2a)}$$

Elastic Theory

To solve this problem by the finite difference method, the arch is divided into 4 segments. The geometric properties of the arch, including the primary bending moment, at the various points are tabulated in the first six columns of Table 1. Since deflection effects are neglected in this type of solution, Eq. 8 becomes $M = M' - Hy$.

TABLE 1

$n = 4$; $h = 2$ ft; $EI = 1.832$ lb-ft^2; $\frac{h^2}{EI} = 2.18 \times 10^{-3}$ lb^{-1}

Sta.	x (ft)	y (ft)	tan θ	sec θ	M' (ft-lb)	M (ft-lb)
0	0	0	0.6250	1.1792	0	0
	1					22.5
1	2	0.9375	0.3125	1.0478	548.40	30.0
	3					22.5
2	4	1.2500	0.0000	1.0000	691.20	0
	5					-22.5
3	6	0.9375	-0.3125	1.0478	488.40	-30.0
	7					-22.5
4	8	0	-0.6250	1.1792	0	0

Since this arch has pinned ends, the boundary conditions are given by Eqs. 5, and Eq. 1 in finite difference form may be written as follows:

At station 1:

$$(\eta_0 - 2\eta_1 + \eta_2) = \frac{h^2}{EI} \sec\theta_1 \, (M_1' - Hy_1)$$

and since $\eta_0 = 0$,

$$(-2\eta_1 + \eta_2) = 2.18 \times 10^{-3} \times 1.0478 (548.40 - 0.9375H)$$

or

$$-2\eta_1 + \eta_2 = (1254.6 - 2.1448H) 10^{-3}$$

At Station 2:

$$(\eta_1 - 2\eta_2 + \eta_3) = 2.18 \times 10^{-3} \times 1.000 (691.20 - 1.250H)$$

or

$$\eta_1 - 2\eta_2 + \eta_3 = (1509.2 - 2.7293H) 10^{-3}$$

At Station 3:

$$(\eta_2 - 2\eta_3 + \eta_4) = 2.18 \times 10^{-3} \times 1.0478 (488.40 - 0.9375H)$$

and since $\eta_4 = 0$,

$$\eta_2 - 2\eta_3 = (1117.4 - 2.1448H) \times 10^{-3} \qquad (10)$$

There are four unknowns η_1, η_2, η_3, and H, but only three equations. It, therefore, is necessary to utilize Eq. 2 together with the boundary conditions to obtain a solution for H. Writing Eq. 2 in finite difference form, and then using Eqs. 5, the following relations are obtained:

At Station 0:

$$\frac{1}{2h}(\xi_1 - \xi_{-1}) = -\frac{1}{2h}(\eta_1 - \eta_{-1}) \times 0.625$$

or $\xi_1 = -0.625\eta_1$

At Station 2:

$$(\xi_3 - \xi_1) = -(\eta_3 - \eta_1) \times 0 \quad \text{or} \quad \xi_3 = \xi_1 \qquad (11)$$

At Station 4:

$$(\xi_5 - \xi_3) = -(\eta_5 - \eta_3) \times (-0.625) \quad \text{or} \quad \xi_3 = 0.625\eta_3$$

and since $\xi_1 = \xi_3$, $\eta_3 = -\eta_1$.

This condition allows the elimination of one unknown from the group of relations, Eqs. 10, formulated from Eq. 1. There are now as many equations as there are unknowns and they may be readily solved for the result H = 552.96 lb. It can be shown that this value is exact, even though a very coarse mesh was used.

Deflection Theory

In this solution the effects of both vertical and horizontal deflections will be included. Accordingly, Eqs. 8 and 9 must be used to formulate the bending moment at every point; the remainder of the solution is identical with the elastic theory solution. However, now let the arch be divided into eight equal segments of length h = 1 ft. The geometric properties of the arch, together with the primary bending moments modified for horizontal deflection effects, are given in Table 2.

Before applying Eq. 1, it is advisable to utilize Eq. 2 to obtain expressions for ξ in terms of η. Thus, utilizing Eqs. 5, the following relations are obtained when Eq. 2 is converted to finite difference form:

At Station 0: $\frac{1}{2h}(\xi_1 - \xi_{-1}) = -\frac{1}{2h}(\eta_1 - \eta_{-1}) \tan\theta_0$

TABLE 2

$n = 8$; $h = 1$ ft; $EI = 1.832$ lb-ft^2; $\frac{h^2}{EI} = 0.54585 \times 10^{-3}$ lb^{-1}

Sta.	x (ft)	y (ft)	tan θ	sec θ	$M' + M_\xi$ (ft-lb)
0	0	0	0.62500	1.1792	0
1	1	0.5469	0.46875	1.1045	$324.90 + 274.20\,\xi_1$
2	2	0.9375	0.31250	1.0478	$548.40 + 172.80\,\xi_2$
3	3	1.1719	0.15625	1.0122	$670.50 + 71.40\,\xi_3$
4	4	1.2500	0	1.000	$691.20 - 30.00\,\xi_4$
5	5	1.1719	-0.15625	1.0122	$625.50 - 101.40\,\xi_5$
6	6	0.9375	-0.31250	1.0478	$488.40 - 172.80\,\xi_6$
7	7	0.5469	-0.46875	1.1045	$279.90 - 244.20\,\xi_7$

or

$$\xi_1 = -0.625\,\eta_1$$

At Station 1:

$$(\xi_2 - \xi_0) = -(\eta_2 - \eta_0) \times 0.46875$$

and since $\xi_0 = 0$, and $\eta_0 = 0$,

$$\xi_2 = -0.46875\,\eta_2$$

At Station 2: $(\xi_3 - \xi_1) = -(\eta_3 - \eta_1) \times 0.3125$

and since $\xi_1 = -0.625\,\eta_1$,

$\xi_3 = -0.3125\,(\eta_3 + \eta_1)$

At Station 3: $(\xi_4 - \xi_2) = -(\eta_4 - \eta_2) \times 0.15625$ (12)

or $\xi_4 = -0.15625\,(\eta_4 + 2\eta_2)$

At Station 8: $(\xi_9 - \xi_7) = -(\eta_9 - \eta_7) \times (-0.625)$

or $$\xi_7 = 0.625\eta_7$$

At Station 7: $(\xi_8 - \xi_6) = -(\eta_8 - \eta_6) \times (-0.46875)$

and since $\xi_8 = 0$, and $\eta_8 = 0$,

$$\xi_6 = 0.46875\eta_6$$

At Station 6: $(\xi_7 - \xi_5) = -(\eta_7 - \eta_5) \times (-0.3125)$

or $$\xi_5 = 0.3125(\eta_7 + \eta_5)$$

Eq. 1 may now be written in finite difference form at the seven interior stations. With the aid of Eqs. 12 this will result in seven equations with seven unknowns, provided it is assumed that H = 552.96 lb. This value of H is based on the elastic theory solution. An example of the conversion of Eq. 1 to an algebraic equation is made below for station 1:

$$(\eta_0 - 2\eta_1 + \eta_2) = \frac{h^2}{EI} \sec\theta_1 \left[M' + M_\xi - H(y + \eta)\right]$$

$$-2\eta_1 + \eta_2 = 0.54585 \times 10^{-3} \times 1.1045 \left[(324.90 + 274.20\xi_1) - 552.96\,(0.5469 + \eta_1)\right]$$

But since from the first of Eqs. 12 $\xi_1 = -0.625\eta_1$, the above equation reduces to

$$-1563.303\eta_1 + 1000\eta_2 = 13.556 \qquad (13)$$

For the remaining six interior points, the procedure for setting up the algebraic equations is similar. The final matrix of coefficients of these equations is shown in Table 3. For this matrix Gauss' Scheme, also known as Doolittle's Method, was used to obtain the solution of the simultaneous equations. The large number of zero terms lying outside of the main diagonal greatly expedite the solution. The unknown ξ terms are determined from Eqs. 12.

The results are assembled in Table 4 and are plotted in Figs. 3, 4, and 5. The values in the last column of Table 4, the internal bending moments, are

obtained from a sum of the primary bending moment, the bending moment contributed by horizontal deflection, and the bending moment caused by the horizontal thrust. For example, at Station 1:

$$M = M' + M_\xi - H(y + \eta)$$

$$= 324.90 + 274.20 \times 0.0391 - 552.96(0.5469 - 0.0626)$$

$$= 324.90 + 10.72 - 267.80$$

$$= +67.82 \text{ ft-lb}$$

Table 3
Matrix of Coefficients for Equation 1
Deflection Theory

η_1	η_2	η_3	η_4	η_5	η_6	η_7		$\frac{h^2}{EI} M \sec\theta$	
-1563.303	1000	0	0	0	0	0		-13.556	
1000	-1637.411	1000	0	0	0	0		-17.158	
+12.328	1000	-1682.156	1000	0	0	0		-12.424	
0	-5.117	1000	-1700.725	1000	0	0	+	0	=0
0	0	0	1000	-1676.976	1000	+17.508		+12.439	
0	0	0	0	1000	-1637.411	1000		+17.159	
0	0	0	0	0	1000	-1574.607		+13.574	

TABLE 4

Sta.	η (ft)	ξ (ft)	M (ft-lb)
1	-0.0626	0.0391	67.82
2	-0.0843	0.0395	83.44
3	-0.0583	0.0378	57.42
4	-0.0005	0.0264	- 0.51
5	0.0569	0.0371	-57.74
6	00.0825	0.0387	-82.31
7	0.0610	0.0381	-65.54

Also plotted on Figs. 3, 4, and 5 are the results of a finite difference solution similar to the above except that mesh of 16 was used. In addition, the results from an analysis using the Newmark procedure[15] are also shown.

It is significant to note that the analysis with a coarse mesh of only 8 subdivisions yields values that are in such close agreement with the answers from more accurate solutions. At the point of maximum moment, there is only a 6 per cent discrepancy between the values from the 8 and the 16 division solutions. However, as may be seen from Fig. 5, the bending moment curve based on the elastic theory is considerably in error. For example, the bending moments at the quarter points are over 100 per cent in error. Qualitatively, this may be an extreme case, but by no means an impossible one.

The fact that the effects of axial strain were neglected does not seem to have altered the accuracy of the final answers, as may be judged by a comparison of analytical and experimental results. As is known, axial strains effects become of importance in the case of the very flat arches. Their inclusion in the finite difference solution presents no serious difficulties. An examination of Eqs. 1 and 2 shows that, with the exception of the term for axial thrust, N, merely numerical constants are contributed to the right hand side of the corresponding equations written in finite difference form. The term N is a function of the vertical shear and horizontal thrust at any point and thus may be expressed in terms of a numerical constant and the unknown thrust H at each point.

CONCLUSIONS

As indicated by the foregoing discussion, the application of finite differences to the differential equations that govern the action of arch ribs makes possible a rapid and accurate solution of these equations. The solution is a completely general one which is not restricted in any manner by variations in external loading or geometrical properties of the arch itself. The degree of accuracy of the results is determined by the fineness of the mesh chosen. It has been shown that suprisingly good results were obtained even with the use of mesh of 8 sub-divisions. The procedure may be used in conjunction with either the elastic or deflection theory of arches. This is dependent upon whether or not the expression for the bending moment is modified initially for effects of horizontal and/or vertical deflections. If the expression is not so modified then solution of the differential equations by the finite difference method is an alternate to the usual methods of determining the redundant forces of indeterminate arches. Actually, in some cases it may be much simpler to apply this method than the customary procedures.

However, it is felt that the finite difference method has its greatest application in the solution of the deflection theory problem. Here, it definitely possesses the advantages of rapidity, accuracy and generality over the other available methods. With the advent of the electronic and digital computers for the large scale inversion of matrices, the finite difference method should prove particularly advantageous for solutions of large arch structures.

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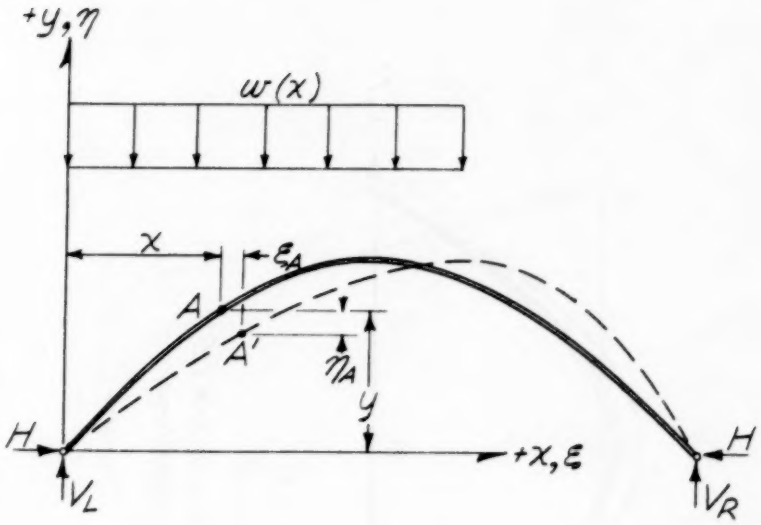
+y, η
w(x)
x
ξA
A
A'
ηA
y
H
+x, ξ
H
VL
VR

FIG. 1

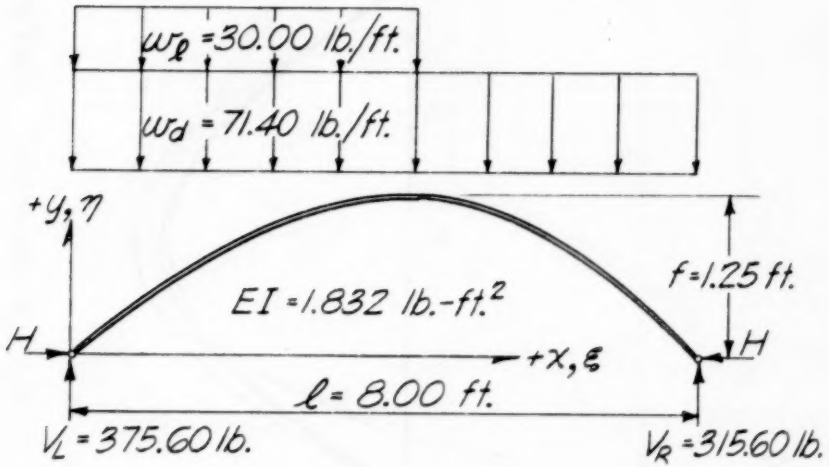
wℓ = 30.00 lb./ft.
wd = 71.40 lb./ft.
+y, η
EI = 1.832 lb.-ft.²
f = 1.25 ft.
H
+x, ξ
H
ℓ = 8.00 ft.
VL = 375.60 lb.
VR = 315.60 lb.

FIG. 2

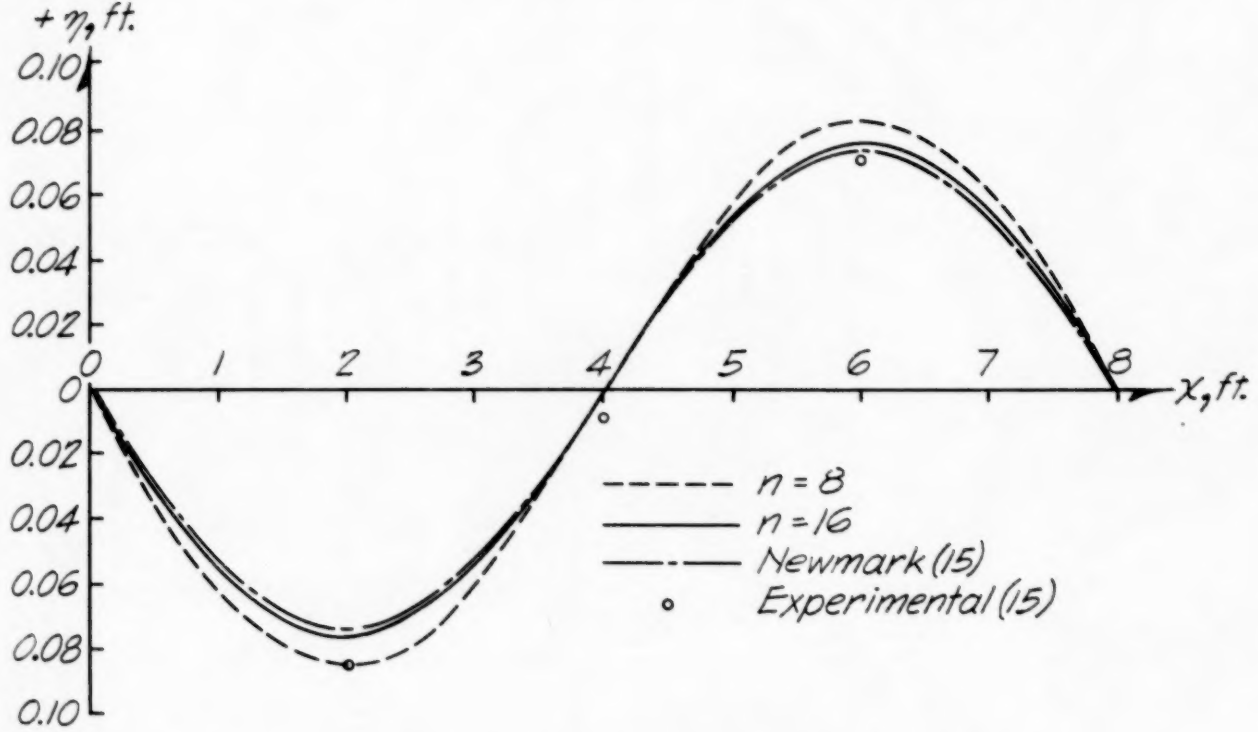

FIG. 3 - Vertical Deflections of Arch Axis

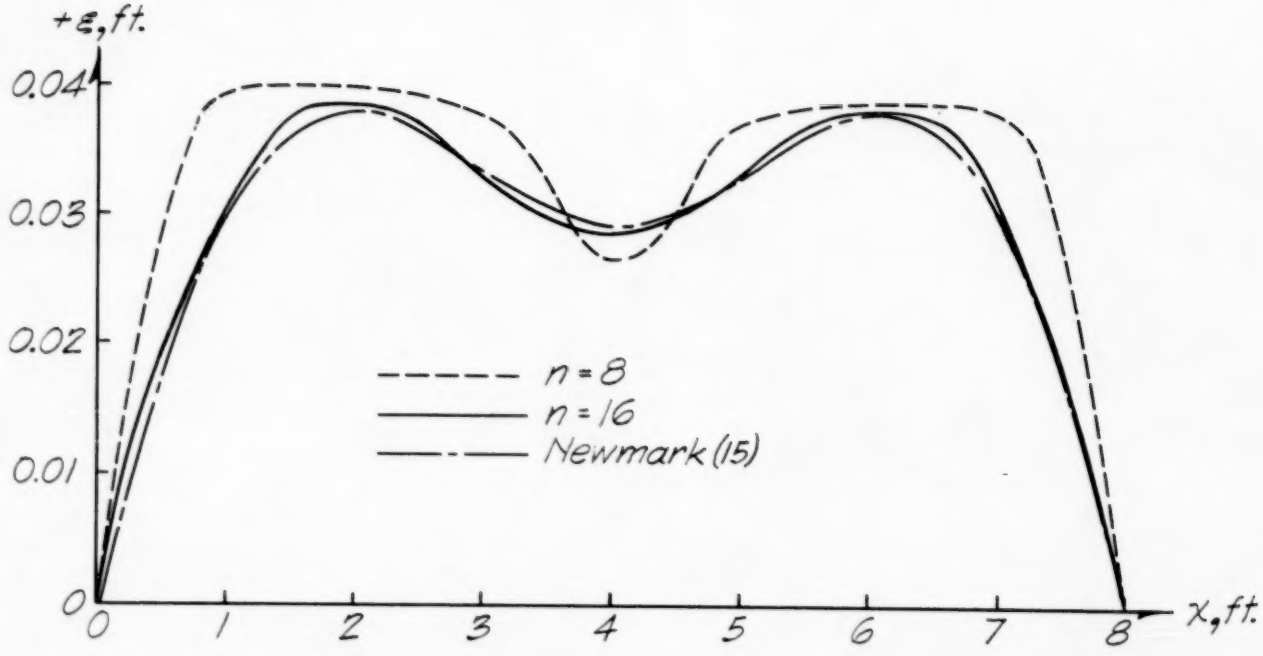

FIG. 4–Horizontal Deflections of Arch Axis

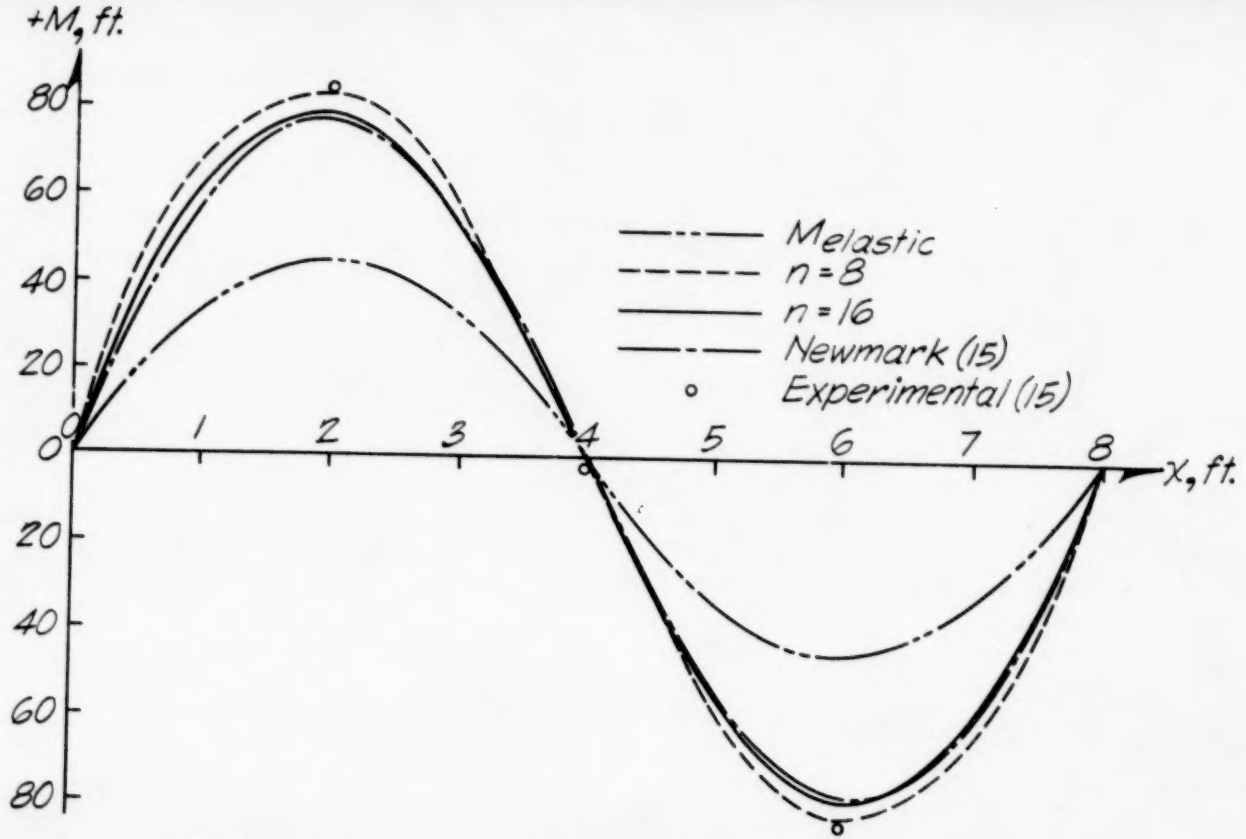

FIG.5-Bending Moment Along Arch Axis